COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 8 (4) 1988 333—343

BOUNDING THE DIAMETER OF DISTANCE. REGULAR GRAPHS

C. D. GODSIL[1]

Received January 30, 1987
Revised September 22, 1987

Let G be a connected distance-regular graph with valency $k>2$ and diameter d, but not a complete multipartite graph. Suppose that θ is an eigenvalue of G with multiplicity m and that $\theta \neq \pm k$. We prove that both d and k are bounded by functions of m. This implies that, if $m>1$ is given, there are only finitely many connected, co-connected distance-regular graphs with an eigenvalue of multiplicity m.

1. Introduction

A graph G is called *distance regular* if, given any two vertices u and v, the number of vertices in G at distance i from u and distance j from v only depends on i, j and the distance between u and v. By taking u equal to v we see that a distance-regular graph is necessarily regular. The main result of this paper is:

1.1 Theorem. *Let G be a connected distance-regular graph of valency k, where k is greater than two. Suppose that G is not complete multipartite and that θ is an eigenvalue of the adjacency matrix of G with multiplicity m. Then if $\theta \neq \pm k$, the diameter of G is at most $3m-4$ and $k \leqq (m-1)(m+2)/2$.*

An important corollary of this is that the number of distance-regular graphs with an eigenvalue θ with multiplicity $m>1$ is finite. Distance-regular graphs are interesting for a number of reasons, including their relation to problems in group theory and finite geometry. A readable introduction to the theory of distance regular graphs is provided in [2]. The restrictions imposed on G, k and θ in our theorem are quite natural. The distance-regular graphs of valency two are just the circuits. These can have arbitrarily large diameter, but their eigenvalues all have multiplicity at most two. It is also known that $-k$ is an eigenvalue of a connected k-regular graph iff it is bipartite, in which case its multiplicity is one. The complement of mK_n has $-n$ as an eigenvalue with multiplicity m. (However since these graphs all have diameter two, their exclusion does not significantly weaken the theorem.)

The idea of bounding the diameter of G in terms of the multiplicity of an eigenvalue may appear surprising. However the problem of deriving such a bound is, in some sense, dual to the much studied problem of bounding the diameter of a distance-regular graph in terms of its valency. The proof of Theorem 1.1 is based on a geometric

[1] This work was supported by NSERC grant A5367.
AMS subject classification code (1980): 05 C 99.

tool which we first used in [5]. The relevant theory will be described in the next section, the proof itself will be found in Section 3. In Section 4 we present some further information concerning graphs with diameter greater than $2m$, including a proof that G can have diameter $3m-4$ iff it is the 1-skeleton of the icosahedron. The final section provides some further consequences of our methods. In particular we see that for any positive integer s there are only finitely many distance-regular graphs with diameter $3m-s$. It is also shown that a distance-regular graph with second largest eigenvalue having multiplicity three is planar.

Throughout this paper we will use I to denote an identity matrix and J to denote a square matrix with all entries equal to one. The orders of these matrices will always be determined by the context. If x and y are vertices in some graph, we will write "$x \sim y$" as an abbreviation for "x is adjacent to y". We call a graph G *co-connected* if its complement $\bar{G}$ is connected. The only distance-regular graphs which are not co-connected are the complete multipartite graphs. (To see these note that if G is not co-connected then it has diameter at most two. Given this, the claim is a routine exercise.)

2. Representations of graphs

Let G be graph with vertex set $\{1, \ldots, n\}$. We denote the adjacency matrix of G by $A = A(G)$. This is the $n \times n$ matrix such that $A_{ij} = 1$ if vertices i and j are adjacent in G, and $A_{ij} = 0$ otherwise. Since A is symmetric, its eigenvalues are real. If θ is an eigenvalue of A with multiplicity m, let U_θ be an $n \times m$ matrix with columns forming an orthonormal basis for the eigenspace associated with θ. Let $u_\theta(i)$ be the i-th row of U_θ. (The existence of U_θ is another consequence of the symmetry of A. The fact that it is not uniquely determined by θ will cause us no problems.)

Thus, given θ, we now have a mapping from $V(G)$ into $\mathbf{R}^m$. From the definition of U_θ we see that $AU_\theta = \theta U_\theta$ and so we have

$$\sum_{j \sim i} u_\theta(j) = \theta u_\theta(i). \tag{1}$$

We will call any mapping from $V(G)$ into $\mathbf{R}^m$ satisfying (1) a representation of G with weight θ. Note that we obtain a representation of G for each eigenvalue of G. The following result is proved (without great difficulty) in [5: Lemma 2.1].

2.1 Lemma. *Let A be the adjacency matrix of the graph G. Then if i and j are any two vertices in G and r is a non-negative integer,* $(A^r)_{ij} = \sum_\lambda \langle u_\lambda(i), u_\lambda(j) \rangle \lambda^r$. ∎

Here the sum is over all distinct eigenvalues λ of A and $\langle , \rangle$ denotes the usual scalar product of two real vectors. The vertices i and j may be equal. Note that $(A^r)_{ij}$ is equal to the number of walks in G from vertex i to vertex j with length r. To apply our theory to distance-regular graphs we need some preliminary information.

Suppose G is a distance-regular graph. We define the i-th distance matrix $A_r = A_r(G)$ of G to be the $n \times n$ 01-matrix with ij-entry equal to 1 iff the distance in G between vertex i and vertex j is r. Thus $A_0 = I$, A_1 is the adjacency matrix of G and $\sum_{i=1}^{n} A_i = J$. The ij-entry of $A_r A_s$ is equal to the number of vertices in G at distance r from i and distance s from j. From the definition of a distance-regular graph it follows that this number only depends on r, s and the distance between i and j. Hence $A_r A_s$ can be written as a linear combination of distance matrices of G. (This implies in addition that $A_r A_s$ is a symmetric matrix and hence that $A_r A_s = A_s A_r$.)

2.2 Lemma. *Let G be a distance-regular graph of diameter d and let θ be an eigenvalue of G. If i and j are two vertices of G then $\langle u_\theta(i), u_\theta(j)\rangle$ is determined by the distance between i and j in G.*

Proof. From our remarks above it follows that there are numbers $c_i(n)$ such that

$$(A_1)^n = \sum_{r=0}^{d} c_r(n) A_r.$$

If i and j are at distance s in G then this implies that the ij-entry of (A_1) is equal to $c_s(n)$. Using 2.1 we then deduce that

$$c_s(n) = \sum_{\lambda} \langle u_\lambda(i), u_\lambda(j)\rangle \lambda^n$$

where, once again, the sum is over the distinct eigenvalues of A. If we fix i and j and take $n=0, \ldots, d$ this gives us a $(d+1)\times(d-1)$ system of equations satisfied by the inner products $\langle u_\lambda(i), u_\lambda(j)\rangle$. This system is non-singular because the matrix of coefficients is a Vandermonde matrix. Therefore our inner products are determined by the eigenvalues of A and the numbers $c_s(m)$. Since the latter only depend on the distance s between i and j, our lemma is proved. ∎

The above result is also proved in Bannai and Ito [1: Lemma II.8.2]. One consequence of it is that, for a fixed eigenvalue θ, the vectors $u_\theta(i)$ all have the same length. Thus when G is distance-regular, u_θ maps $V(G)$ into a sphere in $\mathbf{R}^m$.

3. Bounding the diameter

We will now use the machinery of the previous section to derive our main result. The first lemma details some structural information about distance-regular graphs.

3.1 Lemma. *Let G be a distance-regular graph of diameter d and valency k. If x and y are two vertices in G at distance i, let c_i, a_i and b_i be the number of vertices in G adjacent to y and at distance $i-1$, i and $i+1$ from x respectively. Then:*

(a) $$1 = c_1 \leqq \ldots \leqq c_d$$

(b) $$k = b_0 \geqq \ldots \geqq b_{d-1}$$

(c) *if* $i+j \leqq d$ *then* $c_i \leqq b_j$.

Proof. These are standard results. See, for example, [2] for (a) and (b) and [8] for (c). ∎

3.2 Theorem. *Let G be a distance-regular graph of valency $k \geqq 3$, which is connected and co-connected. Let θ be an eigenvalue of $A(G)$ with multiplicity m. Then if $\theta \neq \pm k$, the diameter of G is at most $3m-4$.*

Proof. We call a set S of vertices of G *independent* if its image $u_\theta(S)$ under u_θ is a linearly independent set of vectors. Clearly any linearly independent set of vertices of G contains at most m elements. The distance between two vertices x and y will be denoted by $\delta(x, y)$. A path in G with end-vertices x and y is *geodetic* if its length is equal to $\delta(x, y)$. Let P be a fixed geodetic path in G with length equal to the diameter d of G. Let x be the first vertex of P and let Q be the longest path with independent vertex set starting at x and contained in P. Denote the length of Q by q. (So $q+1\leqq m$).

The key to our proof is the following observation. If z is a vertex of G such that $u_\theta(z)$ lies in the span of $u_\theta(Q)$, then the vector $u_\theta(z)$ is determined by the sequence of inner products $\langle u_\theta(y), u_\theta(z)\rangle$, $y\in Q$, and hence by the distances $\delta(y, z)$, $y\in Q$. In other words, if z and z' are two vertices of G such that $u_\theta(z)$ and $u_\theta(z')$ lie in the span of $u_\theta(Q)$ and $\delta(y, z)=\delta(y, z')$ for all y in Q then $u_\theta(z)=zu_\theta(z')$. The rest of the proof is divided into a series of steps.

(a) *If u_θ takes equal values on two adjacent vertices of G then $\theta=k$.*
If 0 and 1 are two adjacent vertices in G then, by Lemma 2.2, we find that $\langle u_\theta(x), u_\theta(y)\rangle=\langle u_\theta(0), u_\theta(1)\rangle$ for any pair of adjacent vertices x and y in G. Since G is connected it follows that if $u_\theta(0)$ and $u_\theta(1)$ are equal then u_θ is constant on $V(G)$. As we have

$$\theta u_\theta(0) = \sum_{j\sim 0} u_\theta(j) \tag{1}$$

it follows that $\theta u_\theta(0)=ku_\theta(1)=ku_\theta(0)$. Hence $\theta=k$.

(b) *If 0 and 1 are adjacent vertices in G and $u_\theta(0)=-u_\theta(1)$ then $\theta=-k$.*
If 0 and 1 are adjacent vertices in G and $u_\theta(0)=-u_\theta(1)$ then $u_\theta(x)=-u_\theta(y)$ for every pair of adjacent vertices x and y in G. So (1) implies that $\theta u_\theta(0)=-ku_\theta(0)$ and therefore $\theta=-k$. (Remark: together (a) and (b) imply that $q>0$.)

(c) *If z and z' are vertices in G such that $u_\theta(z)=u_\theta(z')$ then $\delta(z, z')>2$.*
Suppose (0, 1, 2) is a geodetic path in G and $u_\theta(0)=u_\theta(2)$. Then if x and y are any two vertices in G with $\delta(x, y)=2$, we must have $u_\theta(x)=u_\theta(y)$. If G contains an induced odd cycle with at least five vertices then it follows that there must be two adjacent vertices x and y such that $u_\theta(x)=u_\theta(y)$, and consequently we are back in case (b) above. If G is bipartite and u_θ is not constant then it must be constant on the colour classes of G. Then (1) above implies that $\theta u_\theta(0)=ku_\theta(1)$ and, interchanging the roles of 0 and 1, also that $\theta u_\theta(1)=ku_\theta(0)$. Consequently $\theta^2=k^2$ and so $0=\pm k$. Taken with (a), this proves our claim.

The only remaining possibility is that G contains a triangle vwx. If y is adjacent to x but not to v or w then $u_\theta(v)=u_\theta(y)=u_\theta(w)$ and we are again back in (b) (since v and w are adjacent). This shows that no vertex can be adjacent to exactly one of the vertices in a given triangle. Hence every vertex in G adjacent to x must be adjacent to v or w. Therefore the neighbourhood of x induces a connected subgraph of G. From this it follows that any vertex at distance two from x must be adjacent to all the neighbours of x. As G is a regular graph it follows now that it is a complete multipartite graph.

(d) *If P_1 and P_2 are two geodetic paths in G with the same length, their images in $\mathbf{R}^m$ under u_θ are congruent.*
There is an obvious bijection from $V(P_1)$ to $V(P_2)$ which preserves the distance between vertices. Since the distance between any two vertices x and y in G determines the distance between their images $u_\theta(x)$ and $u_\theta(y)$ in $\mathbf{R}^m$, it follows that there is an

orthogonal transformation of $\mathbf{R}^m$ mapping $u_\theta(P_1)$ onto $u_\theta(P_2)$, i.e., these images are congruent.

(e) *If P' is geodetic path containing Q, and with the same initial vertex, then $u_\theta(P')$ is contained in the span of $u_\theta(Q)$.*
Let x and x_q be the two endpoints of Q. Suppose z is the unique vertex in $P' \setminus Q$ adjacent to x_q. Then $Q \cup \{z\}$ is dependent, by our choice of Q. Since $Q \cup \{z\}$ induces a geodetic path in G with $q+2$ vertices, it follows from (d) that the image of any geodetic path with $q+2$ vertices is dependent, being spanned by the first $q+1$ vertices. Now since each subset of $q+2$ consecutive vertices of P' forms a geodetic path, our claim follows by a simple induction argument.

(f) *For all $i > q$ we have $b_i = 1$ and for all $i \leqq d-q$ we have $c_i = 1$.*
Suppose z and z' are two vertices at distance $q+1$ from x and adjacent to x_q. Then, for each vertex y in Q, we have $\delta(y, z) = \delta(y, z')$. If we can verify that both $u_\theta(z)$ and $u_\theta(z')$ lie in the span of $u_\theta(Q)$, it will follow that $u_\theta(z) = u_\theta(z')$. But $Q \cup \{z\}$ and $Q \cup \{z'\}$ induce geodetic paths in G with the same length and so, using (d) and (e), we deduce that both these paths are dependent. Hence both $u_\theta(z)$ and $u_\theta(z')$ lie in the span of $u_\theta(Q)$. Hence $u_\theta(z)$ and $u_\theta(z')$ are equal. Since $\delta(z, z') \leqq 2$, this contradicts (c). Consequently $b_q = 1$ and so, by Lemma 3.1, our claims are proven.

(g) *If $d \geqq 3q$ then $k = 2$.*
We have $k = a_i + b_i + c_i$ for any i and from (f) we now know that c_q and b_q both equal 1 when $d \geqq 2q$. Thus it will suffice to show that $a_q = 0$. Let s_i be a vertex in P at distance $q+i$ from x. Given that $b_{q+i} = 1$ for all $i \geqq 0$, a simple induction argument on i shows that s_i is the unique vertex in G at distance i from x_q and at distance $q+i$ from x. In particular each of the a_i vertices adjacent to s_i and at distance i from x_q are at distance at most $q+i-1$ from x. This implies that $a_i + 1 \leqq c_{q+i}$. If $d \geqq 3q$ then, by (f), $c_{2q} = 1$ and thus $a_q = 0$. ∎

It will prove convenient to have some of the technical consequences of the above argument spelled out in detail.

3.3 Corollary. *Let G be a connected, co-connected distance-regular graph of diameter d, valency k and with an eigenvalue θ of multiplicity m. Assume that any geodetic path in G which is independent with respect to θ has length at most q. Then, if $\theta \neq \pm k$ and $d > q$ we have:*

(a) $$b_i = 1 \quad \text{for} \quad i \geqq q,$$

(b) $$c_i = 1 \quad \text{for} \quad i \leqq d-q,$$

(c) $$a_i + 1 \leqq c_{q+i} \quad \text{for} \quad i \leqq d-q,$$

(d) $$\text{if} \quad k > 2 \quad \text{then} \quad d < 3q.$$

Proof. Here (a) and (b) are immediate consequences of step (f) in the proof of the theorem, while (c) follows from the argument used in step (g). Finally, (d) is just a restatement of (g) itself. ∎

Note that our method provides us with information about G whenever $d > q$. However the only general information we have about q is that $q+1 \leqq m$, and so we can say little when $d+1 \leqq m$.

4. Bounding the valency

In this section we derive some bounds on the valency of a distance-regular graph with an eigenvalue of multiplicity m, in particular when the diameter is large.

4.1 Lemma. *Let G be a connected, co-connected distance-regular graph of diameter d and valency k at least three. Assume that G has an eigenvalue $\theta\neq\pm k$ of multiplicity m. Then $k\leqq\binom{m+1}{2}-1$ and if $d\geqq 2m-1$ then $k\leqq m$.*

Proof. Let x be a vertex in G and let N be the set of vertices in G adjacent to x. Clearly the distance between any two vertices in N is at most two. From step (c) in the proof of Theorem 3.2, it follows that u_θ is injective on N. Since the elements of $u_\theta(N)$ all have the same inner product with $u_\theta(x)$, they form a set of k points on a sphere in $\mathbf{R}^{m-1}$ with the property that the distance between any pair of points takes at most two different values. From [4: Example 4.10] this implies our bound on $|u_\theta(N)|$.

Now assume that $d\geqq 2m-1$. Since $q+1\leqq m$, it follows from Corollary 3.3 that $a_1=0$, hence all pairs of points in $u_\theta(N)$ are the same distance apart. Thus they form a regular simplex with k points in $\mathbf{R}^{m-1}$, and therefore we have $k\leqq m$. ∎

Taken together, Theorem 3.2 and Lemma 4.1 imply Theorem 1.1 in the introduction to this paper. We remark that, in [10], Terwilliger derives some very interesting results related to the above lemma, showing in particular that we can have $k>m$ only in restricted circumstances. His methods are similar in spirit to the ones we are using in this paper.

A distance-regular graph of diameter d is said to be *antipodal* if its vertex set can be partitioned into classes with the property that any two vertices in the same class are at distance d and any two vertices in different classes are at distance less than d. (This interesting class of graphs is discussed in [2: Chapter 22]).

4.2 Lemma. *Let G be a connected, co-connected distance-regular graph of diameter d and valency k at least three. Assume that G has an eigenvalue $\theta\neq\pm k$ of multiplicity m. Let q be the maximal length of a geodetic path in G, independent with respect to θ. If $d=3q-1$ then G is antipodal, and if $d>2q$ then $k\leqq m+1+2q-d$.* ∎

Proof. Assume that $d=3q-1$. We will show that, if x is a vertex in G, there is a unique vertex in G at distance d from x. If $d=3q-1$ then, from Corollary 3.3, we find that $c_i=1$ and $a_i=0$ for $i=1,\ldots,q-1$. Since $a_i+b_i+c_i=k$, this implies that $b_i=k-1$ for $i=1$ to $q-1$. For $i=q,\ldots,2q-1$ we have $b_i=c_i=1$. This in turn implies that $a_i=k-2$ whence, by 3.3(c), we deduce that $c_{2q-1+i}\geqq k-1$ for $i=1,\ldots,q$. Let k_i denote the number of vertices in G at distance i from x. A simple counting argument shows that $k_ib_i=k_{i+1}c_{i+1}$. Using this and the information just derived we find that $k_{3q-1}=1$, and so G is antipodal as claimed.

Let x_{q-r} be a vertex in G at distance $q-r$ from x and let y and z be two vertices at distance $q-r+1$ from x adjacent to x_{q-r}. As $d>2q$ we see from 3.3(b) that $c_i=1$ for $i=2,\ldots,q-r$, whence there is a unique path P in G from x to x_{q-r} with length $q-r$. Since $\delta(u,y)$ and $\delta(u,z)$ are equal for each vertex u in P, it follows that $u_\theta(y)-u_\theta(z)$ is orthogonal to each of the $q-r+1$ vectors $u_\theta(u)$. Since these $q-r+1$ vectors are linearly independent, their orthogonal complement in $\mathbf{R}^m$ is an $(m+r-1-q)$-dimensional space.

Suppose that $d=3q-r$. Then $c_{2q-r}=1$ and therefore $c_{q-r}=1$ and $a_{q-r}=0$. Hence $k=1+b_{q-r}$. If $q>r$ then $d\geqq 2q+1$ and $a_1=0$. Thus, if $d=3q-r$ and $q>r$ then any two vertices adjacent to x_{q-r} and at distance $q+1-r$ from x are at distance two in G. Therefore their images under u_θ form a simplex in $\mathbf{R}^{m+r-1-q}$ Accordingly we must have $b_{q-r}\leqq m+r-q$ and $k\leqq m+r+1-q$.

We note that when G is antipodal as described above, each antipodal set contains exactly two vertices.

4.3 Corollary. *Let G be a connected, co-connected distance-regular graph of diameter d and valency k at least three. If G has an eigenvalue $\theta\neq\pm k$ with multiplicity m and $d=3m-4$ then it is isomorphic to the 1-skeleton if the dodecahedron.*

Proof. If G is complete multipartite then $d=2$, so we must have $m=2$. By 4.2 we see that G is cubic and antipodal. We now inspect the list of cubic distance-regular graphs in [3] and then, by somewhat tedious calculations, deduce that the icosahedron is the only example satisfying our conditions. ∎

We remark that the results in the next section show that if $d=3m-4$ then m is bounded. This means that we could, if necessary, prove 4.3 more directly. In particular we could eliminate the dependence on some of the heavy machinery appealed to in [3], albeit at the cost of some extensive case arguments. Also, if a complete multipartite graph has $d=3m-4$ then m must be equal to 2 and so the graph must $\overline{3K_n}$ for some n.

5. Injective representations

Let G be a distance-regular graph with an eigenvalue θ of multiplicity m. In general, the map u_θ need not be injective, although we have shown that if $u_\theta(i)=u_\theta(j)$ then the distance between i and j is greater than 2. In this section we establish some situations where u_θ is injective and we use this information to extend some of our results from the previous sections.

Assume that P is a geodetic path in G with length equal to the diameter d of G. Suppose that the vertices of P in order are $x_0, \ldots, x_d$ and set $w_i=\langle u_\theta(x_0), u_\theta(x_i)\rangle / \langle u_\theta(x_0), u_\theta(x_0)\rangle$. From equation (1) in Section 2, we see that

$$\theta w_i = c_i w_{i-1}+a_i w_i+b_i w_{i+1} \tag{1}$$

with the understanding that $w_{-1}=w_{d+1}=0$.

5.1 Lemma. *Let the polynomials $p_0(x), \ldots, p_{d+1}(x)$ be defined by the recursion*

$$b_i p_{i+1}(x) = (x-a_i)p_i(x)-c_i p_{i-1}(x), \tag{2}$$

where $p_{-1}=0$, $p_0=1$ and $c_0=b_d=0$. Then $w_i=p_i(\theta)$ for $i=1, \ldots, d$ and $p_{d+1}(x)$ is the minimal polynomial of $A(G)$.

Proof. The first claim is an immediate consequence of (1), and is also proved on page 142 of [1]. From the proof given there is also follows that $p_{d+1}(x)$ vanishes at each of the $d+1$ distinct eigenvalues of A. ∎

Let λ be a real number. It follows from (1) that if $p_i(\lambda)$ and $p_{i+1}(\lambda)$ are both zero for some $i \leqq d$ then $p_i(\lambda)=0$ for $i=0, \dots, d$. Since $p_0=1$, this is impossible. If $p_i(\lambda)$ is equal to zero then, again using (1) and noting that b_i is positive, we must have $p_{i-1}(\lambda)p_{i+1}(\lambda)<0$. Finally, if λ is an eigenvalue of A, we see that $p_{d+1}(\lambda)=0$ and so $p_d(\lambda)\neq 0$. (Hence w_d is never zero.) We say that the sequence $p_0(\lambda), \dots, p_d(\lambda)$ has exactly m sign-changes if there are exactly m indices i in the range $1, \dots, d$ such that either $p_i(\lambda)p_{i+1}(\lambda)<0$ or $p_i(\lambda)=0$ and $p_{i-1}(\lambda)p_{i+1}(\lambda)<0$.

5.2 Lemma. *Let θ be an eigenvalue of G. Then θ is the i-th largest eigenvalue of G iff the sequence $(p_0(\theta), \dots, p_d(\theta))$ has exactly $i-1$ sign-changes.*

Proof. The polynomials $p_i(x)$, $i=0, \dots, d+1$, form a Sturm sequence on the reals. (These are discussed at length in many places. One convenient reference is [6: §6.3].) Consequently the number of sign changes in the sequence $(p_0(\theta), \dots, p_d(\theta))$ is equal to the number of zeroes of p_{d+1} which are greater than θ. ∎

A connected distance regular-graph G of diameter d is said to be *primitive* if, for each $i=1, \dots, d$, the graph G_r with adjacency matrix A_r is connected. If G is not primitive then it is *imprimitive*. It is known that if G is imprimitive then either G_2 is not connected and G is bipartite, or else $G_1, \dots, G_{d-1}$ are connected but G_d is not. In the latter case it follows that $V(G)$ can be partitioned into classes with the property that any two vertices in the same class are at distance d while two vertices in different classes are at distance less than d. (A discussion of these matters for distance-transitive graphs will be found in [2: Chapter 22]. The arguments used there can easily be extended to distance-regular graphs.)

5.3 Lemma. *Let G be a connected distance-regular graph of diameter d and valency k at least three, with an eigenvalue $\theta\neq\pm k$. If the number of eigenvalues of $A(G)$ greater than θ is odd then u_0 is an injective function.*

Proof. Under the hypothesis on θ, the number of sign-changes in the sequence $(w_0, \dots$ $\dots, w_d)$ must be odd. This implies in particular that w_d must be negative. To prove the lemma it will suffice to show that if $w_i=w_0$ then $i=0$. Suppose that for some $i>0$ we do have $w_i=w_0$. It follows that if x and y are any two vertices in G at distance i then $u_\theta(x)=u_\theta(y)$. Let G_r be the graph with adjacency matrix $A(G_r)=A_r$. It is immediate that if x and y lie in the same component of G_r then $u_\theta(x)=u_\theta(y)$. Thus if G_r is connected then u_θ is constant on $V(G)$ and so our sequence $(w_0, \dots, w_d)$ has no sign-changes. Thus we may assume that G_r is not connected. From our discussion of imprimitive distance-regular graphs above we may thus assume that either $r=d$ or $r=2$. If $r=d$ then $w_d=1$, which is impossible since w_d is negative. If $r=2$ then u_θ takes only two values on $V(G)$. If $r=2$ then, using (c) in the proof of Theorem 3.2, we deduce that $\theta=\pm k$. ∎

The relation between primitivity and injectivity can also be derived using the results in Section II.9 of Bannai and Ito [1]. In the case when θ is the second largest eigenvalue of G, Lemma 5.2 and the second part of Lemma 5.3 have also been obtained independently by D. Powers [7]. The arguments just used can be extended to show that a distance-regular graph with valency k is primitive iff u_θ is injective for all eigenvalues θ of G not equal to $\pm k$. This characterisation of primitivity will not be of any use to us here, and so no further details will be given.

Even when u_θ is injective on $V(G)$, it does not necessarily follow that the sequence $w_0, \ldots, w_d$ is nonincreasing. In particular, the images of the vertices adjacent to a given vertex x need not be the points in $u_\theta(V(G)\setminus x)$ closest to $u_\theta(x)$. There is, however, one important case where this does hold true.

5.4 Lemma. *Let G be a distance-regular graph with valency k and diameter d at least two. If θ is the second largest eigenvalue of G and $x \in V(G)$ then the points in $u_\theta(V(G))$ closest to $u_\theta(x)$ are the images of the vertices adjacent to x.*

Proof. Using the recurrence in Lemma 5.1 we compute that

$$p_0 = 1, \quad p_1(x) = x/k, \quad p_2(x) = (x^2 - a_1 x - k)/kb_1.$$

(In determining p_2 we used the fact that $c_1 = 1$, as noted in 3.1(a).) The remainder of the argument is broken up into a number of separate steps.

(a) *The second largest eigenvalue of G is non-negative.*

We recall first that if K is obtained from the graph H by deleting a vertex then the eigenvalues of K interlace the eigenvalues of H, whence it follows that H has at least as many non-negative eigenvalues as K. Since G has diameter at least two, it contains an induced subgraph isomorphic to the path on three vertices. This path has two non-negative eigenvalues (namely 0 and $\sqrt{2}$) and therefore G has at least two non-negative eigenvalues. Since the largest eigenvalue k of G is simple, our claim is proved.

(b) *If $w_{i-1} \geqq w_i \geqq 0$ then $w_i > w_{i+1}$ and if $w_{i-1} \leqq w_i \leqq 0$ then $w_i < w_{i+1}$.*

We have $\theta w_i = b_i w_{i+1} + a_i w_i + c_i w_{i-1}$. Since $b_i + a_i + c_i = k$, it follows that $k^{-1}\theta w_i$ is a convex combination of the numbers w_{i-1}, w_i and w_{i+1}. It follows that

$$\min\{w_{i-1}, w_i, w_{i+1}\} \leqq k^{-1}\theta w_i \leqq \max\{w_{i-1}, w_i, w_{i+1}\}.$$

Since $k^{-1}\theta < 1$ our claims follow immediately.

(c) *If $-1 \leqq \theta \leqq k$ then $w_1 \geqq w_2$.*

We see easily that $p_1(x) - p_2(x)$ is equal to $(x^2 - (a_1 + b_1)x - k)/kb_1$. As $c_1 = 1$ it follows that $a_1 + b_1 = k - 1$, which implies that $p_1(x) - p_2(x) = (x+1)(x-k)/kb_1$. Consequently $p_1(\theta) \leqq p_2(\theta)$ whenever θ lies between -1 and k.

The proof can now be completed. Let θ be the second largest eigenvalue of G. By (a) above we infer that $0 \leqq \theta < k$. From 5.2 we see that the sequence $(w_0, \ldots, w_d)$ has exactly one sign change and so, in particular, $w_d < 0$. By (c) we see that $w_1 \geqq w_2$ and it therefore follows from (b) above that this sequence decreases monotonically, at least until the first negative element is reached. As u_θ is injective (by 5.3), the lemma follows immeadiately. ∎

5.5 Corollary. *If G is a connected distance-regular graph with the second largest eigenvalue θ having multiplicity three then G is planar.* ∎

Corollary 5.5 can be strengthened. Thus if θ has multiplicity $m = 4$ then the neighbourhood of a vertex must be planar, and more complicated statements can be made for general m. We leave their formulation as an exercise to the interested reader. The next result is an alternative bound on the diameter of G in terms of m. Terwilliger has derived a lower bound for m in terms of the valency and girth of G in [9], using similar methods. (His bound has much the same form as that implied by our equation (3) below.)

5.6 Theorem. *Let G be distance-regular graph of diameter d and valency k with an eigenvalue θ of multiplicaty m. Assume that G is connected and co-connected. If* $\theta \neq \pm k$ *then*

$$\left\lfloor \frac{d}{2} \right\rfloor \leqq m - \log m / \log (k-1).$$

Proof. Let x be fixed vertex in G and let N_i be set of vertices in G at distance i from x. Let r be a positive integer and assume that the girth of G is at least $4r+2$. Then the vertices in N_r fall into $|N_{r-1}|$ classes such that two vertices in the same class are at distance two in G, while vertices in different classes are at distance $2s$, for some $s \geqq 2$. (Of course, the classes are just the sets of vertices in N_r adjacent to a given vertex in N_{r-1}.) From the proof of Lemma 5.3 we see that if $i > 0$ and $w_0 = w_i$ then $i = d$. Since the diameter of G is at least half its girth, we see that u_θ is injective on N_r. We will derive our bound on d by determining a lower bound on the rank of $u_\theta(N_r)$.

Consider the vector $z = u_\theta(a) - u_\theta(b)$, where a and b are two vertices in N_r with a common neighbour in N_{r-1}. If c is a vertex in N_r, not adjacent to the common neighbour of a and b, then $u_\theta(c)$ is orthogonal to z (since a and b are equidistant from c). The span of $u_\theta(N_r)$ is the orthogonal sum of $|N_{r-1}|$ subspaces, each of which has dimension equal to the dimension of the space U spanned by the vectors $z = u_\theta(a) - u_\theta(b)$, where a and b range over the $k-1$ vertices in N_r adjacent to y. It remains to determine the dimension of U. However, since the girth of G is at least 4, the image of the vertices adjacent to y is a regular simplex, and therefore the dimension of U is $k-2$. This implies immediately that $u_\theta(N_r)$ spans a subspace with dimension at least $(k-2)|N_{r-1}|$.

We can now complete the proof. From Corollary 3.3 we see that if $d \geqq 2m-2+2r$ then $a_i = 0$ and $c_i = 1$ for $i = 1, \ldots, 2r$ and so the girth of G is at least $4r+2$. In this case we also find that $|N_{r-1}| = k(k-1)^{r-2}$. Accordingly we must have

$$m \geqq k(k-1)^{r-2}(k-2) = (k-1)^r - (k-1)^{r-2}. \tag{3}$$

If we divide both sides of this expression by $(k-1)^{r+1}$ and take logarithms, we obtain

$$(\log m + (k-1)^{-2}) / \log (k-1) \geqq r+1.$$

(Here we have used the fact that $\log(1-(k-1)^{-2}) \geqq (k-1)^{-2}$.) Consequently we deduce that

$$\left\lfloor \frac{d}{2} \right\rfloor \leqq m + \log m / \log (k-1),$$

which is the bound given in the statement of the theorem.

The bound just derived seems reasonable, but is almost certainly not best possible. It is, at least, strong enough to yield the following result. ∎

5.7 Corollary. *For any fixed positive integer s, there are only finitely many connected, co-connected distance regular graphs with an eigenvalue of multiplicity* $m > 1$ *and diameter* $3m - s$. ∎

In conclusion we remark that this work raises a number of questions. One problem currently under investigation is that of determining the distance-regular graphs with eigenvalues of low multiplicity. (For mutiplicity three, they are the 1-skeletons of the Platonic solids, and the complete multipartite graphs $\overline{4K_n}$.)

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C. D. GODSIL

Department of Mathematics and Statistics
Simon Fraser University
Burnaby, B.C.
Canada V5A 1S6

Current address:

Combinatorics and Optimization,
University of Waterloo
Waterloo, Ontario,
Canada N2L 3G1